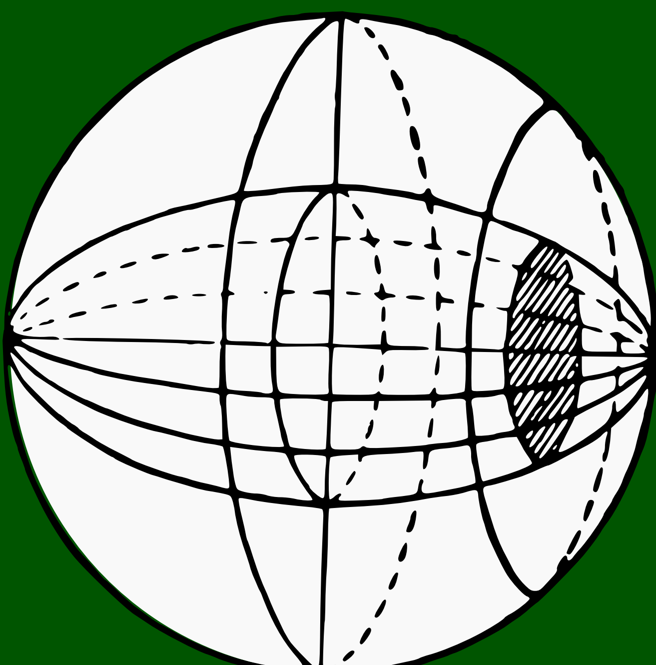
RUSSIAN TRACTS ON ADVANCED MATHEMATICS AND PHYSICS
VOULUME IX

I. Natanson

SUMMATION OF INFINITESIMAL QUANTITIES



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SUMMATION OF INFINITESIMAL QUANTITIES

by

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PREFACE

The study of Integral Calculus is difficult enough. This subject has reached its modern form as a result of mutual interlacing of large number of highly hetrogeneous concept.

However, the fundamental concept of Integral Calculus (even according to ancient concepts) is the *notion of the limit of the sum of an indefinitely increasing number of limitlessly diminishing terms*, which is very simple and natural. It can be mastered without much efforts.

This concept is found very useful in solving series of important problems of geometry and physics and assimilating the idea of limit more deeply. In a sense, this concept, therefore, serves as an important tool in the study of higher mathematics.

In the present text an account has been given of the above concept and its application to various concrete problems. The material contained in this text forms a supplementary and enlarged treatment of the lectures which I extensively delivered at various courses.

April 13, 1953.
Leningrad.

I. Natanson.

CONTENTS

CHAPTER I

1 SOME FORMULAE OF ALGEBRA

1. We shall need later certain algebraic, formulae which are not always dealt in elementary works. These formulae give sums of the type

$$S_p = 1^p + 2^p + 3^p + \ldots + n^p,$$

where p stands for an integral number. We require the expression for the sum S_p only for small values of p:

$$p = 1, 2, 3.$$

Let us find the required sum.

2. **Sum of the members of natural series.**

First of all, we shall find the sum:

$$S_1 = 1 + 2 + 3 + \ldots + n.$$

This is the sum of n members of the arithmetical progression with the first member $a_1 = 1$ and the difference $d = 1$; its value may be determined with the help of :

$$S_1 = \frac{n\,(n+1)}{2}. \tag{1}$$

We shall employ another method of establishing the formula (1), although a bit complicated, but successfully applicable

to any sum S_p.

Let us take the important identity

$$(n+1)^2 = n^2 + 2n + 1$$

and substitute in it successively $n-1$, $n-2$, for n and so on, till we reach unity, As a result, we shall get the series of equations:

$$\left.\begin{aligned}(n+1)^2 &= n^2 + 2n + 1\\ n^2 &= (n-1)^2 + 2(n-1) + 1\\ (n-1)^2 &= (n-2)^2 + 2(n-2) + 1\\ &\dots\dots\dots\dots\\ 2^2 &= 1^2 + 2 \cdot 1 + 1\end{aligned}\right\} \qquad (2)$$

Summing up all these equations and noting that the column of the terms on the left-hand side consist of almost only those terms, which form also the column of the first terms on the right-hand-side. The distinction between these columns is that on the left-hand-side term 1^2 is absent which is the last term of the column on the right-hand-side, and the term $(n+1)^2$ is present which is not there in the right-hand-side.

It is clear, that after cancelling out similar terms of both the columns, we get:

$$(n+1)^2 = 1^2 + \{2n + 2(n-1) + \dots + 2 \cdot 1\} + \\ + \{1 + 1 + \dots + 1\}.$$

The number of terms in the second bracket is equal to the number of equations (2), *i. e.* equal to n, so that, the whole of this bracket is also equal to n. Further we notice that if we take out the common factor 2, from the first bracket, we are

left in this bracket with the sum S_1. If we also replace 1^2 by 1, we find

$$(n+1)^2 = 1 + 2S_1 + n.$$

Hence

$$2S_1 = (n+1)^2 - (n+1) = (n+1)\,[(n+1) - 1] = n\,(n+1),$$

and finally

$$S_1 = \frac{n\,(n+1)}{2},$$

so that we again get the formula (1).

3. Sum of Squares. We shall now apply the above method for finding out the value of the sum of squares of first n natural numbers, *i. e.* the sum

$$S_2 = 1^2 + 2^2 + 3^2 + \ldots + n^2.$$

For this we substitute successively in the equality

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1$$

$n-1$, $n-2$ and so on for n till we reach unity. This will lead us to the series of equalities:

$$\left.\begin{aligned}(n+1)^3 &= n^3 + 3n^2 + 3n + 1\\ n^3 &= (n-1)^3 + 3\,(n-1)^2 + 3\,(n-1) + 1\\ (n-1)^3 &= (n-2)^3 + 3\,(n-2)^2 + 3(n-2) + 1\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ 2^3 &= 1^3 + 3\,.\,1^2 + 3\,.\,1 + 1\end{aligned}\right\} \quad (3)$$

We shall add all these equalities. As in the previous case we can make considerable simplification here also; viz. from the column of the terms on the left-hand-side all the terms disappear, except first *i. e.* $(n+1)^3$, and from the column of the first terms on the right-hand-side all the terms vanish except the last *i. e.* 1^3.

Further, if we take out the common factor 3 from the column of the second terms of the right-hand-side then. it is evident we are left with the sum S_2 which is to be found. Exactly in the same way, the column of the third terms of the right-hand-side gives three times the sum S_1 already found above. If we notice further, that the number of lines in (3) is equal to n, we find:

$$(n+1)^3 = 1^3 + 3S_2 + 3S_1 + n.$$

Now replace 1^3 by 1, and S_1 by the expression (1), which gives

$$(n+1)^3 = 1 + 3S_2 + 3\,\frac{n(n+1)}{2} + n.$$

Hence

$$3S_2 = (n+1)^3 - \frac{3}{2}\,n(n+1) - (n+1),$$

or $3S_2 = (n+1)\left[(n+1)^2 - \frac{3}{2}\,n - 1\right] = n(n+1)\left(n + \frac{1}{2}\right).$

Therefore

$$3S_2 = \frac{n(n+1)(2n+1)}{2}.$$

Finally we get:

$$S_2 = \frac{n(n+1)(2n+1)}{6}. \tag{4}$$

4. Sum of Cubes. Exactly in the same way, proceeding from the equality

$$(n+1)^4 = n^4 + 4n^3 + 6n^2 + 4n + 1,$$

we get the system of equalities

$$\begin{aligned}
(n+1)^4 &= n^4 + 4n^3 + 6n^2 + 4n + 1,\\
n^4 &= (n-1)^4 + 4(n-1)^3 + 6(n-1)^2 + 4(n-1) + 1,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
2^4 &= 1^4 + 4\cdot 1^3 + 6\cdot 1^2 + 4\cdot 1 + 1.
\end{aligned}$$

After addition and simplification we find:

$$(n+1)^4 = 1 + 4S_3 + 6S_2 + 4S_1 + n.$$

Substituting the values of the sums S_1 and S_2 whose expressions have already been found in (1) and (4) and after performing all the necessary calculations, which undoubtedly the reader can do, we get the expression for the sum S_3 as

$$S_3 = \frac{n^2(n+1)^2}{4}. \tag{5}$$

In a similar manner, the sums S_4, S_5 etc. can be found.

5. Although this still does not have any direct bearing on the subject of this text, we cannot afford to leave a very interesting consequence of the formulae (1) and (5) *i. e.* by the comparision of these formulae, it is clear that

$$S_3 = S_1^2$$

that is

$$1^3 + 2^3 + \dots + n^3 = (1+2+\dots+n)^2. \tag{6}$$

for instance,

$$1^3 + 2^3 = 9 \text{ and } (1+2)^2 = 9,$$

or

$$1^3 + 2^3 + 3^3 = 36 \text{ and } (1+2+3)^2 = 36,$$

or

$$1^3 + 2^3 + 3^3 + 4^3 = 100 \text{ and } (1+2+3+4)^2 = 100.$$

The equality (6) is more interesting, because we can verify that it does not hold good in its generalised form

$$a^3 + b^3 + \dots + k^3 = (a + b + \dots + k)^2$$

for arbitrary numbers $a, b, \dots, k$. For example

$$2^3 + 4^3 = 72, \ (2+4)^2 = 36,$$

and $72 \neq 36$.

6. Symbol Σ. Formulae (1), (4) and (5) may also be written in another form by using the symbol Σ which is widely used in mathematics *i. e.* if there is a series of terms denoted

by one and the same letter say a, distinguished between one another by the suffixes in this letter: $a_1 + a_2 + a_3 + \dots + a_n$, the sum of these terms is denoted by the symbol

$$\sum_{k=1}^{n} a_k, \tag{7}$$

where the letters a_k, show that the characteristic term of this sum is a, with some suffix, and the signs below and above the symbol of summation Σ show, that the suffix in the letter a, covers all the values from 1 to n. This symbol Σ is the Greek Capital letter *Sigma*. With the help of the symbol Σ the sum S_1, S_2, S_3 may be depicted as :

$$S_1 = \sum_{k=1}^{n} k, \quad S_2 = \sum_{k=1}^{n} k^2, \quad S_3 = \sum_{k=1}^{n} k^3,$$

and the formulae (1), (4), and (5) take the form* :

$$\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}, \tag{8}$$

$$\boxed{\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}} \tag{9}$$

$$\boxed{\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}}. \tag{10}$$

* We assume that the reader of this text reads it with a "pencil in his hand". If so, then we recommend him to write out the formulae (8), (9), (10) on a separate sheet and always keep it in front of him.

7. Some characteristics of the Symbol Σ.

Let us note some of the properties of the symbol Σ.

1) If each of the terms is itself a sum of two terms, then their sum also *breaks up into* two sums.

Therefore

$$\sum_{k=1}^{n}(a_k+b_k) = \sum_{k=1}^{n}a_k+\sum_{k=1}^{n}b_k. \qquad (11)$$

To give the proof of the equality (11) it is enough to write its left-hand-ssde in the expanded form

$$(a_1+b_1) + (a_2 + b_2) + \ldots + (a_n+b_n),$$

which, evidently can be rewritten as:

$$(o_1+a_2+ \ldots +a_n) + (b_1+b_2+ \ldots +b_n),$$

and this is the right-hand-side of the equality (11).

2) If all the terms of the sum have a common factor, then it can be taken out before the symbol of the sum

$$\sum_{k=1}^{n}ac_k = c\sum_{k=1}^{n}a_k. \qquad (12)$$

3) If all the terms a_k are equal to one and the same value a, then the sum is equal to this value, multiplied by the number of terms,

$$\sum_{k=1}^{n} a = na. \qquad (13)$$

This property can also be proved easily by the reader. In view of the extraordinary simplicity of the properties of the symbol Σ as shown above, we shall use them subseqently without making a specific reference to them.

CHAPTER II

§2. DETERMINATION OF PRESSURE ON A VERTICAL WALL

8.° Pressure on the wall of a reservoir. Let us assume, that there is a rectangular reservoir full of water; its dimensions are shown in figure 1. We want to find the pressure* P of the water on the front wall of the reservoir.

To find the solution, we have to recollect some of the laws of Hydrostatics

9.° If there is a small horizontal plate in water, then the pressure of water on it, is equal to the weight of the column of water pressing the plate *i. e.* the cylindrical column which forms the plate as its base and the depth of immersion of the plate as its height. Because we are dealing with water, the specific gravity of which is unity, the weight of the above mentioned column is equal to its volume *i. e.* equal to the area of the plate multiplied by the depth of its immersion. This product also gives, the pressure on the horizontal plate. If the plate is not horizontal, then its points are found at different depths and we cannot talk about the depth of immersion of the whole plate.

* Here as well as onwards, while talking of pressure, we mean the total thurst with which the water is pressing the wall, and not the force which is pressing a unit area.

But if the plate is very small, then, approximately we can consider, that all its points are immersed at one and the same depth and let us call this depth as the depth of immersion of the whole plate.

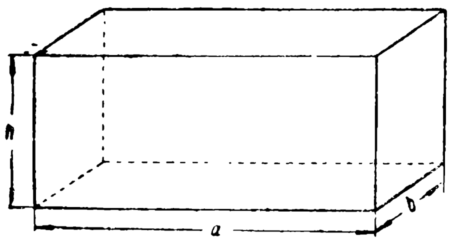

Fig. 1.

Suppose that we are given a very small plate immersed in water ; we shall now define the term 'pressure' on it. To do this let us imagine that we turn this plate about one of its axis so that it becomes horizontal. Since pressure inside the liquid in all directions is equal, and the dimensions of the plate are very small, the operation of turning, does not alter the pressure on the plate appreciably. The above method of finding the pressure is applicable to the plate in its new horizontal position. This process of turning the plate neither changes the area, nor the depth of immersion (since the area of the plate is very small). We can express this statement as : the pressure on the plate in water is equal to the area of the *plate multiplied by the depth of immersion.*

This rule is not exact but is only approximate. The smaller the area of the plate the better is the approximation.

10. After establishing this rule, we shall turn to our problem. The front wall of the reservoir is not very small, and therefore, the established rule will not be directly applicable. In order to employ this rule, we proceed

as follows :

Let us take a very large number n and divide the wall into n equal strips (Fig. 2.) of width $\frac{1}{n}h$ each.

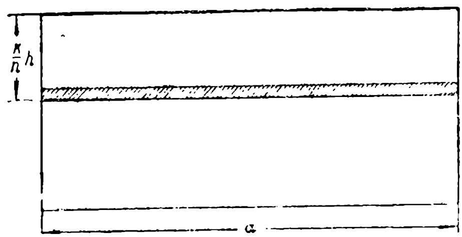

Fig. 2.

Let us now consider one of these elementry strips, for example, the k-th from above. It is very narrow and we can approximately consider that all the points on it lie at the same depth. The pressure on it is found with the help of the rule of article 9°.*

The area of the strip is equal to the product of its length a and its width $\frac{1}{n}h$ *i.e.* $\frac{1}{n}ah$. In order to find the pressure, we should multiply this number by the depth of immersion of the strip.

For the k-th strip from above, the depth is equal to

*) Considering the conclusion of the rule of finding pressure, we see, it is neccessary for exactness that all the points of the strip should lie (although approximately) on equal depths. Therefore, we shall apply the rule to this narrow horizontal strip, although, fortunately, its length cannot be considered to be very small.

$\frac{k}{n}h$*. Hence the pressure P_k on the k-th strip is equal to

$$P_k = \frac{ah^2}{n^2}k.$$

In order to determine the pressure P on the whole wall, we should add the pressures on separate strips, which gives

$$P = \sum_{k=1}^{n} \frac{ah^2}{n^2}k, \quad \text{or} \quad P = \frac{ah^2}{n^2}\sum_{k=1}^{n} k.$$

Making use of the formula (8), we can determine the pressure P as :

$$P = \frac{ah^2}{n^2} \cdot \frac{n(n+1)}{2},$$

or as :

$$P = \frac{ah^2}{2}\left(1 + \frac{1}{n}\right),$$

hence, finally,

$$P = \frac{ah^2}{2} + \frac{ah^2}{2} \cdot \frac{1}{n}. \tag{14}$$

However, the expression found for the pressure is not fully correct. As a matter of fact although the strips taken are very narrow, the different points lie at different depths.

We often use the following symbol in mathematics to denote that two quantities are approximately equal :

* The quantity $\frac{k}{n}h$ is the depth of the lower edge of the strip, but since we neglect the distinction between the depths of separate points of a strip, we take this depth as the depth of immersions of the whole strip. In what follows, we shall repeatedly come across similar conditions.

$$A \doteq B,$$

i.e. by placing a dot on the symbol of equality. Therefore, we should rewrite the relation (14) in the form

$$P \doteq \frac{ah^2}{2} + \frac{ah^2}{2} \cdot \frac{1}{n}. \qquad (14')$$

Moreover, it is clear that more the strips we take *i.e.* larger is the number n, the more exact is the equality. Hence as we increase the number n, we shall get from (14′) more and more exact expressions from the pressure P. Thus, the *exact* pressure appears as the limit* to which the equality

$$\frac{ah^2}{2} + \frac{ah^2}{2} \cdot \frac{1}{n}$$

approaches, when n increases indefinitely. But it is intuitively clear that by increasing n, the number $\frac{1}{n}$, as also $\frac{ah^2}{2} \cdot \frac{1}{n}$ become less and less, tending towards zero. Therefore, the limit of the quantity $\frac{ah^2}{2} + \frac{ah^2}{2} \cdot \frac{1}{n}$ reduces to the first term $\frac{ah^2}{2}$, which also gives us the exact expression for pressure :

$$P = \frac{ah^2}{2}.$$

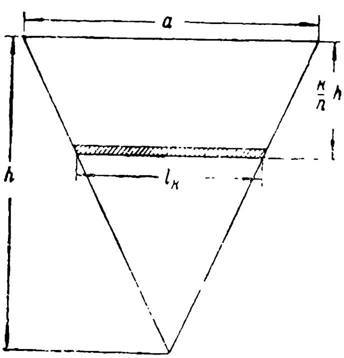

Fig. 3.

11° Pressure on a triangular sheet. We now come across another problem of the same kind.

For instance, we shall try to determine the pressure of water

* We shall remember, that the limit of a variable quantity x_n is a constant number l, such that the absolute value of the difference $x_n - l$ for all sufficiently large values of n is less than any pre-assigned positive number.

on a triangular sheet, lowered vertically in water in such a way that the base of the triangle lies in level with the free surface of the liquid (Fig. 3.)

In order to solve this problem, we proceed from the considerations, already stated in the previous para, *i. e.* we break the sheet into n very narrow horizontal strips.—"elementry strips"—of width $\frac{1}{n}h$ each and we determine the whole pressure as the sum of pressures on separate strips.

Let us take the k-th strip from the top. Neglecting the width of the strip we may consider, that all its points lie at the same depth, equal to $\frac{k}{n} \cdot h$. The pressure P_k on the k-th strip is obtained by multiplying this depth by the area of the strip. This area can be found since it is a trapezium. But, evidently, for the narrow strip, it may be considered with a great degree of preciseness that its shape is rectangular. It simplifies the determination of the area. It is true that there is a little error but this error is less perceptible, the more the number of strips. We already know by the previous example, that the above error does not affect the final result. Here we come in conflict with the ideas constantly employed in solving various kinds of problems. In the calculation of the elementary terms, we should pay attention mainly to the simplicity of its expression neglecting parts of a term which are insignificantly small as compared to those which we take into consideration. With the help of the theory of limits, this principle could have been expressed in a more exact and rigorous form, which we however, do not do, since the examples considered sufficiently elucidate the principles involved.

Taking the k-th strip as a rectangle, we get the area as a product of its length and width. Evidently the width is $\frac{1}{n}h$ and the length l_k (symbol k signifies, that we are dealing with the k-th strip) is found, as is clear from figure 3, by the similarity of triangles *i.e.*

$$l_k : a = \left(h - \frac{k}{n}h\right) : h, \quad \text{hence } l_k = \left(1 - \frac{k}{n}\right)a.$$

Thus the area of the strip is,

$$\left(1 - \frac{k}{n}\right)a \cdot \frac{1}{n}h,$$

and the pressure on it

$$P_k \doteq \frac{ah^2}{n^2}\left(1 - \frac{k}{n}\right)k.$$

The total pressure is found by the summation of the above calculated quantities

$$P \doteq \sum_{k=1}^{n} \frac{ah^2}{n^2}\left(1 - \frac{k}{n}\right)k.$$

$$\text{or} \quad P \doteq \frac{ah^2}{n^2}\sum_{k=1}^{n} k - \frac{ah^2}{n^3}\sum_{k=1}^{n} k^2.$$

Making use of the formulae (8) and (9), we may write the expression in the form :

$$P \doteq \frac{ah^2}{n^2} \cdot \frac{n(n+1)}{2} - \frac{ah^2}{n^3} \cdot \frac{n(n+1)(2n+1)}{6},$$

or

$$P \doteq \frac{ah^2}{2}\left(1 + \frac{1}{n}\right) - \frac{ah^2}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right).$$

This expression for the pressure is approximate, the larger is the number n the more exact it is. It means that for determining the actual pressure, it is necessary to increase n indefinitely in the right-hand-side of this equality and to the find limit of the right-hand-side. Since by increasing n, the fraction $\frac{1}{n}$ approaches zero, the factors $\left(1+\frac{1}{n}\right)$ and $\left(2+\frac{1}{n}\right)$ will approach 1 and 2 respectively. Therefore, (on the basis of the theorem on the limit of the product and difference) the whole of the above expression has the limit $\frac{ah^2}{2}-\frac{ah^2}{6}\cdot 2.$

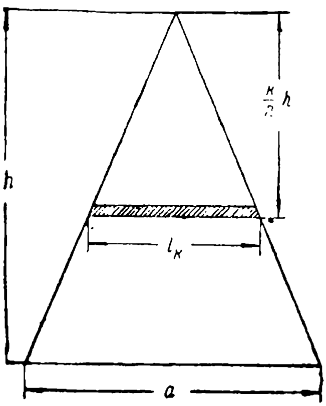

Fig. 4.

Hence

$$P=\frac{ah^2}{2}-\frac{ah^2}{3}$$
$$=\frac{ah^2}{6}.$$

This is the exact value of the pressure.

12°. Let us find the pressure on a verticle sheet of the same shape, but immersed in water so that its vertex lies in level with the surface of water and the base parallel to the surface (Fig. 4).

Dividing the sheet into horizontal strips of width $\frac{1}{n}h$ and taking each such strip as a rectangle, we find the length of the k-th strip by the similarity of triangles ;

$$l_k : a=\frac{k}{n}h : h, \quad \text{hence} \quad l_k=\frac{k}{n}a.$$

The area of the strip is equal to $\frac{k}{n^2}ah$, and since its depth is $\frac{k}{n}h$, the pressure on the k-th strip is equal to

$$P_k \doteq \frac{k^2}{n^3}ah^2.$$

The total pressure is found as the sum of all the P_k

$$P \doteq \sum_{k=1}^{n} \frac{k^2}{n^3}ah^2 = \frac{ah^2}{n^3}\sum_{k=1}^{n} k^2.$$

With the help of the formula (9) we can write P as :

$$P \doteq \frac{ah^2}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

or as :

$$P \doteq \frac{ah^2}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right).$$

The exact expression is found by the limiting operation when n increases indefinitely; to find the limit we repeat the argument given at the end of article 11°. Not going into details we notice that we find the limit, by discarding the term $\frac{1}{n}$ from the brackets, which finally gives

$$P = \frac{ah^2}{3}.$$

13°. Pressure on a semi-circle. In the examples, considered so for the total pressume was calculated by breaking it into a number of separate terms P_k. The calculation of one such term by the simplified method (*i.e.* neglecting the difference in the depths of separate points of one strip and assuming the strip to be in the rectangular form) was easily accomplished. All such separate pressures were summed up

and the limit of the sum involved was found when n is increased indefinitely. To find the limit of the sums we used the formulae (8) and (9) of §1. However, it would be wrong to think that the solution of a problem by this method always leads to the simple sums of §1. On the contrary, we very often meet with far more complicated sums. We illustrate this by an example. For instance, we shall try to determine the pressure on a semi-circular sheet, Fig. 5, placed vertically in water, the diameter of the semi-circle lying in the free surface of the liquid.

Making use of the method already given, divide the sheet into strips of width $\frac{1}{n} R$ where R is the radius of the semi-circle.

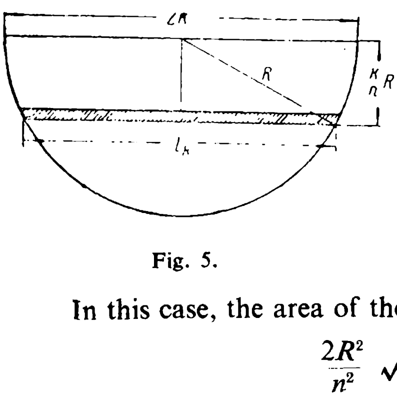

Fig. 5.

Here also, we treat every strip as a rectangle. Its length is found by the Pythagorus Theorem.

$$l_k = 2\sqrt{R^2 - \frac{k^2}{n^2} R^2} = \frac{2R}{n} \sqrt{n^2 - k^2}.$$

In this case, the area of the strip is

$$\frac{2R^2}{n^2} \sqrt{n^2 - k^2},$$

and the pressure on the k-th strip is equal to

$$P_k \doteq \frac{2R^3}{n^3} k \sqrt{n^2 - k^2}.$$

The approximate expression for the total pressure is the sum

$$P \doteq \sum_{k=1}^{n} \frac{2R^3}{n^3} k \sqrt{n^2 - k^2}, \text{ or } P \doteq \frac{2R^3}{n^3} \sum_{k=1}^{n} k \sqrt{n^2 - k^2},$$

Its exact value is the limit of the sum when n increases indefinitely. As a matter of fact, we are not only interested in the sum itself but also in its limit.

Hence

$$P = 2R^3 \lim \left[\frac{1}{n^3} \sum_{k=1}^{n} k \sqrt{n^2 - k^2} \right], \tag{15}$$

where the symbol 'lim' signifies limit.

Therefore, the problem would be solved, if we are able to find the limit

$$\lim \left[\frac{1}{n^3} \sum_{k=1}^{n} k \sqrt{n^2 - k^2} \right]. \tag{16}$$

However we cannot find this limit by the methods already adopted and therefore we cannot solve the given problem. In article 23°, we shall give a method for calculating the limit (16) and solve the present problem.

CHAPTER III

§3. DETERMINATION OF WORK DONE IN PUMPING OUT WATER FROM A VESSEL.

14°. Pumping out water from a cylindrical vessel. In this paragraph, we shall consider the types of problems related to other branches of physics, the solutions of which are obtained with the help of the method of division into indefinitely increasing number of diminishing terms, or, what are called infinitesimal terms.

We shall consider the problem by a typical example. Let us take a cylindrical copper vessel containing water (Fig. 6). Let us assume we empty it with the help of a pump. It is required to find the work done in pumping out all the water.

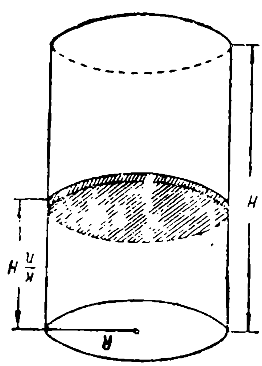

Fig. 6.

We recall that by the work involved in moving a portion of water, we mean the product of the whole force applied to it and the distance travelled by it. Reverting to our problem, we shall notice that for pumping out a portion of the liquid from the vessel, it is sufficient to raise it to the edge of the vessel, and then allow it to overflow under the force of its own weight.

Thus the problem reduces to finding the work, required to raise all the liquid to the level of the edge of the vessel.

It is evident, each portion of the liquid covers a distace equal to its depth below the edge of the vessel. Since the force which is overcome in raising a portion is the weight of the portion, the work done in raising a portion is equal to the product of its weight and its depth. Since we are concerned with water whose specific gravity is equal to unity, the weight of a portion is numerically equal to its volume and, therefore, *the work done in raising a portion of water is equal to the product of its volume and its depth below the edge the of vessel.*

Since different portions of water are at different depths in the vessel, we cannot directly apply the above rule for finding the work done.

In order to make use of this rule, we shall proceed on the lines we solved the problems in the previous articles *i. e.* divide the height H of the cylinder (Fig. 6) into n parts of length $\frac{1}{n}H$ each and draw planes through the points of division parallel to the base of the cylinder. These planes divide the thickness of water into "their" layers. We can approximately consider that all the points of the liquid in a layer lie at one and the same depth. Therefore, using this rule, we can determine the work done in raising one such layer.

The volume of the layer is the volume of the cylinder with radius R (where R is the radius of the vessel) and height $\frac{1}{n}H$, so that it is equal to

$$\pi R^2 \frac{H}{n}.$$

If we deal with k-th layer from above, the work done in raising the k-the layer is equal to

$$T_k \doteq \pi R^2 H^2 \frac{k}{n^2}$$

This equality is approximate and not exact because even in the portion of one layer, the depths of different parts are not equal.

Since the whole work done T is found by adding the above expressions, for all the layers

$$T \doteq \sum_{k=1}^{n} \pi R^2 H^2 \frac{k}{n^2}, \qquad \text{or} \qquad T \doteq \pi R^2 H^2 \cdot \frac{1}{n^2} \sum_{k=1}^{n} k.$$

On the basis of the formula (8), we have

$$T \doteq \pi R^2 H^2 \cdot \frac{1}{n^2} \cdot \frac{n(n+1)}{2},$$

or

$$T \doteq \frac{\pi R^2 H^2}{2} \left(1 + \frac{1}{n}\right). \tag{17}$$

It is easy to see, that by increasing the number n, the exactness of this approximate equality increases. Therefore, we get the exact expression for work by finding the limit of the right-hand-side of (17) by increasing n indefinitly.

This limit, it is evident, is found by deleting the fraction $\frac{1}{n}$, which finally gives

$$T = \frac{\pi R^2 H^2}{2}$$

If we use the expression for the volume of the cylinder $V = \pi R^2 H$, then the given value may be written as :

$$T = V \frac{H}{2}.$$

In other words the work done is equal to that required to raise the entire volume of water to half the height of the vessel.

Note : The last statement may also be obtained with the help of the following considerations and without all the calculations. It is very clear that the work done in removing the middle (*i. e.* one found at the depth of $\frac{1}{2}$ H) layer is equal to its volume multiplied by $\frac{1}{2}$ H. It is possible to find two layers at equal distances from the middle and on either side of it. If the distances of these layers from the middle layer be equal to 'd', then one of them is to be raised to the height $\frac{1}{2}$ $H + d$ and the other to the height $\frac{1}{2}$ $H - d$. Therefore, if each of such layer has the volume V, then the sum of the work done in removing them from the vessel is equal to

$$V\left(\frac{1}{2}\ H + d\right) + V\left(\frac{1}{2}\ H - d\right) = V\text{H}.$$

This means that the work done in removing the above mentioned pairs of layers does not change if both the layers are placed at a depth $\frac{1}{2}$ H. In other words, we may assume that all the water lies at a depth $\frac{1}{2}$ H, and the formula

$$T = \frac{1}{2}\ VH$$

becomes quite evident.

15. Pumping out water from a hollow cone. In this example, we shall consider the analogous problem of finding

the work done in pumping out water from a conical vessal (Fig. 7)

As in the previous example, we shall divide the whole mass of water into n layers each of the thickness $\frac{1}{n}H$. The infinitesimal work done is equal to the depth of the layer multiplied by its volume. This volume is the volume of the truncated cone. However it is very easy to calculate it, treating the layer as a cylinder. This is evidently not correct, but it simplifies calculations. As in article 11°, it is evident that by increasing the number n, the error introduced decreases.

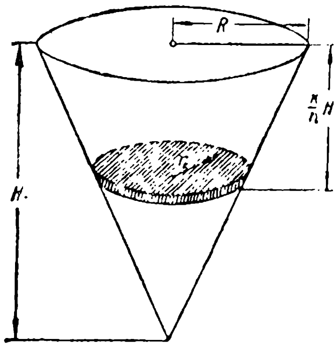

Fig. 7-

Denoting the radius of the k-th layer as r_k, we find that its volume is

$$\pi r_k^2 \frac{1}{n} H.$$

Since the depth of the layer is $\frac{k}{n} H$, the infinitesimal work done is equal to

$$T_k \doteq \pi r_k^2 H^2 \frac{k}{n^2}$$

In this expression the quantity r_k occurs ; we shall express this quantity by the dimensions of the cone. By the similarity of triangles, we have ;

$$r_k : R = \left(H - \frac{k}{n} H\right) : H,$$

hence

$$r_k = \left(1 - \frac{k}{n}\right) R.$$

Substituting this value in the expression of the elementry work, we get

$$T_k \doteq \pi R^2 H^2 \left(1 - \frac{k}{n}\right)^2 \frac{k}{n^2}$$

The tolal work done is equal to

$$T \doteq \sum_{k=1}^{n} \pi R^2 H^2 \left(1 - \frac{k}{n}\right)^2 \frac{k}{n^2},$$

or

$$T \doteq \pi R^2 H^2 \left[\frac{1}{n^2} \sum_{k=1}^{n} k - \frac{2}{n^3} \sum_{k=1}^{n} k^2 + \frac{1}{n^4} \sum_{k=1}^{n} k^3 \right].$$

Making use of the formulae (8), (9), (10) we express the given equality in the form

$$T \doteq \pi R^2 H^2 \left[\frac{1}{2} \left(1 + \frac{1}{n}\right) - \frac{1}{3} \left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + \right.$$
$$\left. + \frac{1}{4} \left(1 + \frac{1}{n}\right)^2 \right].$$

This expression is only approximate, since the layers are not exactly cylindrical and the depth of different points of each layer is different. However increasing the number n, indefinitely, and taking the limit of the right-hand-side, we shall find the exact expression for the work done

$$T = \pi R^2 H^2 \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right)$$

or, finally,

$$T = \frac{1}{12} \pi R^2 H^2.$$

If we remember *) that the volume of the cone is equal to

$$V = \frac{1}{3} \pi R^2 H,$$

then the above expression may be written in the form:

$$T = V . \frac{H}{4},$$

i. e it shows that it is equal to the work done, in raising the entire valume of water to one-fourth the height of the cone.

16° Pumping out water from a hemi-spherical vessel. We shall solve one more example of the same type *i. e.* we shall find the work which is necessary to pump out water from a vessel in the shape of a hemisphere.

Proceeding as before we divide the mass of water into n horizontal layers each of thickness $\frac{1}{n} R$. Assuming each such layer as a cylinder of radius r_k (while dealing with the k-th layer), we see that its volume is

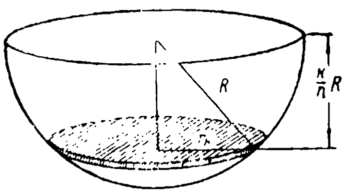

Fig. 8

$$V_k \doteq \pi r_k^2 \frac{1}{n} R,$$

and, consequently, the work done in raising the k-th layer is

$$T_k \doteq \pi r_k^2 \frac{k}{n^2} R^2$$

Now we shall express the radius r_k of the kth layer in terms of the radius R of the sphere. It is obvious from Fig. 8 by using the theorem of Pythogoras

*) However, this formula has been established in article 18.°

$$r_k^2 = R^2 - \left(\frac{k}{n} R\right)^2.$$

Hence

$$T_k \doteq \pi R^4 \left(1 - \frac{k^2}{n^2}\right) \frac{k}{n^2}.$$

The total work is found by the summation of all such terms :

$$T \doteq \sum_{k=1}^{n} \pi R^4 \left(1 - \frac{k^2}{n^2}\right) \frac{k}{n^2},$$

or

$$T \doteq \pi R^4 \left[\frac{1}{n^2} \sum_{k=1}^{n} k - \frac{1}{n^4} \sum_{k=1}^{n} k^3\right],$$

using the formulae of Chapter 1, we have

$$T \doteq \pi R^4 \left[\frac{1}{n^2} \cdot \frac{n\,(n+1)}{2} - \frac{1}{n^4} \cdot \frac{n^2\,(n+1)^2}{4}\right],$$

or

$$T \doteq \pi R^4 \left[\frac{1}{2}\left(1 + \frac{1}{n}\right) - \frac{1}{4}\left(1 + \frac{1}{n}\right)^2\right].$$

This approximate expression for work can be made exact if we neglect $\frac{1}{n}$, (in the limit n goes to in finety and we get gives)

$$T = \pi R^4 \left(\frac{1}{2} - \frac{1}{4}\right)$$

or finally,

$$T = \frac{1}{4} \pi R^4.$$

17°. Pumping out water from a trough. In this concluding article, we shall consider the problem of finding out the work done in pumping out water from a trough *i. e.* from a vessel in the shape of a semi-cylinder (Fig. 9).

Applying the method of dividing into infinitesimal quantities, we divide the whole mass of water into

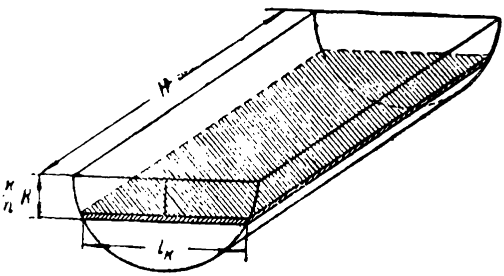

Fig. 9.

n narrow horizontal layers of the form of ractangular slabs (Fig. 9). The volume of one such slab is

$$V_k \doteq H l_k \frac{R}{n},$$

where l_k denotes its width. This width l_k by the theorem of Pythogoras (the chord of the circle is at a distance $\frac{k}{n} R$ from the centre of the circle) is equal to

$$l_k = 2\sqrt{R^2 - \left(\frac{k}{n} R\right)^2},$$

so that the volume of the slab is equal to

$$V_k \doteq 2R^2H \frac{1}{n^2} \sqrt{n^2 - k^2}.$$

Hence the work done in pumping out water of k-th layer is

$$T_k \doteq 2R^3H \frac{k}{n^3} \sqrt{n^2 - k^2},$$

and the total work done is equal to

$$T \doteq \sum_{k=1}^{n} 2R^3H \frac{k}{n^3} \sqrt{n^2 - k^2},$$

or

$$T \doteq 2R^3H \frac{1}{n^3} \sum_{k=1}^{n} k\sqrt{n^2 - k^2}. \tag{18}$$

The above expression is, however, only approximate. For finding out the exact value of the work-done, n should be increased indefinitely and in the limit the right-hand-side of the equality

$$T = 2R^3H. \lim \left[\frac{1}{n^3} \sum_{k=1}^{n} k \sqrt{n^2 - k^2} \right]. \tag{19}$$

Thus our problem reduces to finding the limit

$$\lim \left[\frac{1}{n^3} \cdot \sum_{k=1}^{n} k \sqrt{n^2 - k^2} \right] \tag{20}$$

We see that this limit is the same as that of (16) of article 13°. At present we cannot find this limit, and therefore, the solution of these two physical problems cannot be solved completely. As it has already been pointed out in article 13°, we shall find this limit in article 23°, and solve both the problems.

CHAPTER IV

§ 4. DETERMINATION OF VOLUMES.

18.° The volume of a cone. The methods developed above, find wide applications for solving different kinds of geometrical problems. In the present chapter, we shall show the application of these methods in finding the volumes of different bodies (*).

First of all, we shall handle the problem of finding the *volume of a cone*. To solve this problem, we divide (Fig. 10) the height of the cone into n parts each of length $\frac{1}{n}H$ and draw planes through the points of division, parallel to the base of the cone. These planes cut the whole cone into n layers (which are as a matter of fact, truncated cones) each of which is a cylinder. This in the wider. not precise, sense means that the difference is more or less imperceptible.

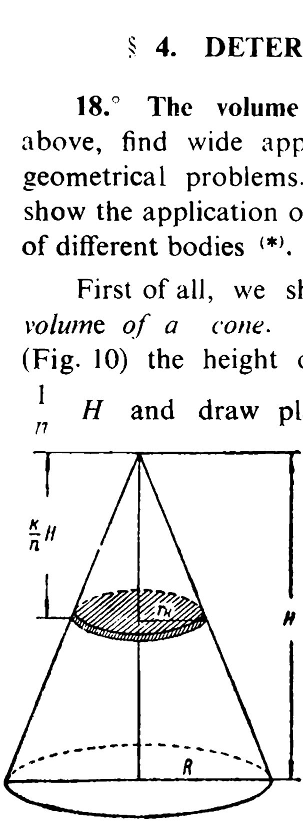

Fig. 10.

Denoting the radius of the k-th layer by r_k, we find that the volume of this cylinder is equal to

*) Since we are interested in the main method, the calculation part of the problem is *neat*, we don't define here the meaning of volume. As it is well knowns, for this definition, it is necessary to make use of the concept of limit

$$V_k \doteq \pi r_k^2 \frac{H}{n}.$$

By the similarity of triangles, we have

$$r_k : R = \frac{k}{n} H : H,$$

hence

$$r_k \doteq \frac{k}{n} R,$$

and the expression for the volume of the k-th layer takes the form :

$$V_k \doteq \pi R^2 H \frac{k^2}{n^3},$$

so that the total volume is equal to

$$V \doteq \sum_{k=1}^{n} \pi R^2 H \frac{k^2}{n^3},$$

or

$$V \doteq \pi R^2 H . \frac{1}{n^3} \sum_{k=1}^{n} k^2,$$

which by the formula (9) assumes the form

$$V \doteq \pi R^2 H \frac{n(n+1)(2n+1)}{6n^3},$$

or

$$V \doteq \pi R^2 H \frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6}. \tag{21}$$

This value of the volume is not exact and is only approximate, for, as already stated, the separate layers, as a matter of fact, are not cylindrical. However, larger the number n, more exact is the above expression, so that the true

value of V is the limit of the right-hand-side of the equality (21) where n increases indefinitely. This limit, evidently, is derived from (21) by neglceting the fraction $\frac{1}{n}$, so that

$$V = \pi R^2 H.\ \frac{1.2}{6}$$

or, finally,

$$V = \frac{1}{3}\pi R^2 H$$

Thus, the volume of a cone is equal to one third of the product of the area of its base and its height.

19. The volume of a pyramid.° Similar argument is permissible to find the volume of a pyramid. We shall consider (Fig. 11) the pyramid of height H, the area of whose base is F. Dividing the height into n equal parts and drawing through the points of division, planes, parallel to the base we divide the pyramid into n prismatic slabs each of thickness $\frac{1}{n}H$ (strictly speaking, these slabs are not prismatic and are truncated pyramids, but as in the above case, we can consider them nearly prismatic).

If the area of the k-th slab from top be F_k, then it is easy to see that

$$F_k : F = k^2 : n^2$$

so that

$$F_k = \frac{k^2}{n^2} F,$$

and consequently, volume of the k-th slab is equal to

$$V_k \doteq F_k.\frac{H}{n} \doteq \frac{k^2}{n^3} FH.$$

The volume of the pyramid is equal to the sum of all such volumes :

$$V = \frac{FH}{n^3}\sum_{k=1}^{n} k^2.$$

or [on the basis of the formula (9)]

$$V = \frac{FH}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right).$$

Increasing n indefinitely and taking the limit of the right-hand-side we find :

$$V = \frac{1}{3}\,FH$$

so that, analogous to the volume of the cone, *the volume of the pyramid is equal to one third of the product of the area of the base and its height.*

20 . Volume of a sphere. We shall now find the volume of a sphere. It is evident that the problem would be solved if we confine ourselves to the case of a hemisphere and after that double the result. Dividing the hemisphere (Fig. 12) into n layers each of thickness $\frac{1}{n}R$ and taking these layers as cylinders. If the radius of the k-th layer is r_k, then its volume, *i. e.* the volume of truncated cylinder, is equal to

$$V_k = \pi r_k^2 \frac{R}{n}.$$

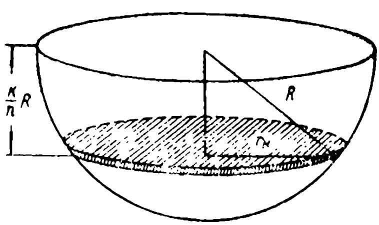

Fig. 12

By the theorem of Pythogoras

$$r_k^2 = R^2 - \frac{k^2}{n^2}\,R^2,$$

so that the expression for the infinitesimal volume takes the form

$$V_k \doteq \pi R^3 \left(1 - \frac{k^2}{n^2}\right) \frac{1}{n},$$

and the volume V^* of the whole hemisphere is the sum of all such V_k's

$$V^* \doteq \pi R^3 \left[\sum_{k=1}^{n} \frac{1}{n} - \frac{1}{n^3} \sum_{k=1}^{n} k^2 \right],$$

which on the basis of the results given in chapter 1, is equal to

$$V^* \doteq \pi R^3 \cdot \frac{6 - \left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)}{6}.$$

The limit of this expression, by increasing n indefinitely gives the exact value of the volume of the hemisphere

$$V^* = \frac{2}{3} \pi R^3.$$

Hence the volume of the whole sphere is

$$V = \frac{4}{3} \pi R^3$$

21°. Volume of the common part of two cylinders. Now we shall solve a difficult problem. We shall consider two cylinders of the same radius whose axis intersect at right angles (Fig. 13). We are to find the volume of the common part of both the cylinders. The crux of the problem lies in a clear conception of what the body looks like. However, we can

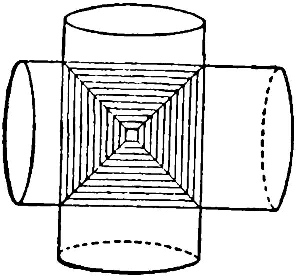

Fig. 13.

solve this question even without conceiving what the body looks like.

To solve this problem, we shall imagine a plane, passing through the axis of both the cylinders; call it the axial plane. This plane (if we assume it to be coinciding with the plane of the diagram) divides the body into two halves, the front and the rear. We shall restrict ourselves to the study of one of them, for instance, the front one, since they are evidently identical.

Now imagine a plane parallel to the axial plane. It will cut each of the cylinders into a strip. These strips in both the cylinders, have the same width. Therefore, the given body when intersected by the plane, gives a *square*.

Having established this it is easy to solve the problem. Draw a perpendicular from the point of intersection of the axes of the cylinders to the axial plane. The length of the segment contained in the first half of the given body is equal to R. Divide the segment into n parts and draw planes through the points of division, parallel to the axial-plane. These planes cut the first half of the given body into n square strips of thickness $\frac{1}{n}R$ each.

It is easy to see from Figure 14, that the side of the k-th square of the given body is equal to

$$l_k = 2\sqrt{R^2 - \left(\frac{k}{n}R\right)^2}$$

and therefore, its area is

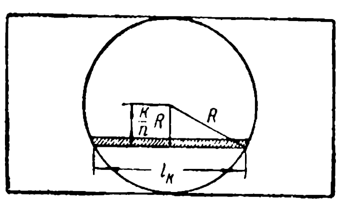

Fig. 14.

$$l_k^2 = 4\,R^2\left(1 - \frac{k^2}{n^2}\right),$$

and the volume of the k-th strip is

$$V_k \doteq l_k^2 \cdot \frac{R}{n} = 4R^3\left(1 - \frac{k^2}{n^2}\right)\frac{1}{n}.$$

The volume V^* of the entire first half of the body is the sum of all V_k's, *i. e.*

$$V^* \doteq \sum_{k=1}^{n} 4R^3\left(1 - \frac{k^2}{n^2}\right)\frac{1}{n},$$

or

$$V^* \doteq 4R^3\left[\sum_{k=1}^{n}\frac{1}{n} - \frac{1}{n^3}\sum_{k=2}^{n}k^2\right],$$

i.e.

$$V^* \doteq 4R^3\left[1 - \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\right].$$

This approximate equality becomes exact if we indefinitely increase n.

Thus, the volume of the first half of the body is equal to

$$V^* = \frac{8}{3}\,R^3.$$

The whole volume V, is therefore

$$V = \frac{16}{3}\,R^3,$$

which is the required result.

It is very curious that inspite of the very complicated character of the body its volume has been found very easily.

22.° Volume of a cylindrical segment. We shall now consider the so called "cylindrical segment"—a body, cut from a cylinder by a plane passing through a diameter of its base (Fig. 15). Let $AB = H$, its base $OA = R$. We shall express the volume of the segment in terms of H and R.

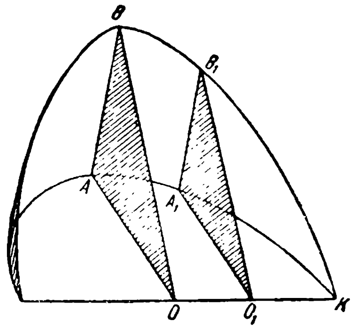

Fig. 15

To solve this problem let us divide the radius Ok into n equal parts and draw planes through the points of division parallel to the plane of the triangle OAB. These planes cut one of the two halves of the cylindrical segment into n triangular strips each of thickness $\frac{1}{n} R$. One of these strips is $O_1A_1B_1$ and is shown in the figure.

Let us consider the volume of the strip assuming it to be prismatic.

Let $O_1A_1B_1$ be such a strip, so that

$$OO_1 = \frac{k}{n} R.$$

By the theorem of Pythogoras it is easy to show that

$$O_1A_1 = \sqrt{OA_1^2 - OO_1^2},$$

or, in other words

$$O_1A_1 = R\sqrt{1 - \frac{k^2}{n^2}}.$$

Further, by the similarity of the triangles OBA and $O_1A_1B_1$, we have,

$$A_1B_1 : AB = O_1A_1 : OA,$$

or

$$A_1B_1 : H = R\sqrt{1 - \frac{k^2}{n^2}} : R.$$

Hence

$$A_1B_1 = H\sqrt{1 - \frac{k^2}{n^2}}.$$

The area of the triangle $O_1A_1B_1$ is given by

$$\frac{1}{2} RH\left(1 - \frac{k^2}{n^2}\right).$$

The volume of the k-th strip is found by multiplying this area with the thickness of the strip *i. e.* $\frac{1}{n} R$. It means then, the infinitesimal volume is

$$V_k \doteq \frac{1}{2} R^2H\left[\frac{1}{n} - \frac{k^2}{n^3}\right]$$

and the volume V^* of the whole half of the segment is

$$V^* \doteq \frac{1}{2} R^2H\left(\sum_{k=1}^{n}\frac{1}{n} - \sum_{k=1}^{n}\frac{k^2}{n^3}\right),$$

or

$$V^* \doteq \frac{1}{2} R^2H\left[1 - \frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6}\right].$$

The limit of this expression gives the exact value of the half the volume of the segment

$$V^* = \frac{1}{3} R^2H.$$

Hence the volume of the whole segment is equal to

$$V = \frac{2}{3} R^2 H \tag{22}$$

23°. Another Method. We shall attempt this very problem in another way, *viz.* we shall divide the radius OA into n parts (Fig. 16) and draw planes through the points of division perpendicular to this radius. They will cut the whole cylindrical segment into n rectangular strips as shaded in the figure.

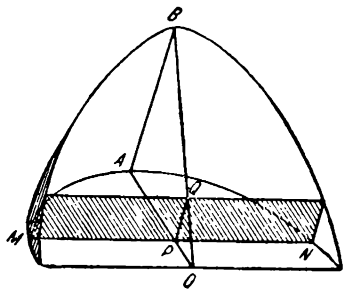

Fig. 16.

We shall, find the volume of the k-th strip (we shall assume them to be prismatic). The thickness of each of them is $\frac{1}{n} R$.

Let the k-th strip be as shaded in the figure.

We get
$$OP = \frac{k}{n} R.$$

In this case by the theorem of Pythogoras the chord MN is given by

$$MN = 2\sqrt{R^2 - \frac{k^2}{n^2} R^2}\,.$$

By the similarity of triangles OPQ and OAB, we have

$$PQ : OP = AB : OA,$$

or

$$PQ : \frac{k}{n} R = H : R,$$

so that

$$PQ = \frac{k}{n} H,$$

and the area of the rectangle being equal to $PQ.MN$ is

$$2RH \frac{k}{n} \sqrt{1 - \frac{k^2}{n^2}}$$

Therefore, the elementry volume is equal to

$$V_k \doteq 2R^2H. \frac{k}{n^3} \sqrt{n^2-k^2}.$$

Hence the whole volume is equal to the sum :

$$V \doteq 2R^2H. \frac{1}{n^3} \sum_{k=1}^{n} k \sqrt{n^2-k^2}.$$

However, this equality is only approximate, and we shall get the exact expression from it if we replace the right-hand-side by its limit when n increases indefinitely. This gives

$$V = 2R^2H \lim \left[\frac{1}{n^3} \sum_{k=1}^{n} k \sqrt{n^2-k^2}\right] \tag{23}$$

Once again, for the third time we come across the limit

$$\lim \left[\frac{1}{n^3} \sum_{k=1}^{n} k \sqrt{n^2-k^2}\right] .$$

As we do not know the above limit we cannot solve the problem by this method. On the other hand, comparing the expressions (22) and (23), we find the value of the required limit, *i. e*. cancelling $2R^2H$, we get

$$\lim \left[\frac{1}{n^3} \sum_{k-1}^{n} k \sqrt{n^2-k^2}\right] = \frac{1}{3}, \tag{24}$$

and hence, we finally establish the limit.

Putting the above limit in the equality (15) of article 13° we find the required pressure

$$P = \frac{2}{3} R^3.$$

Exactly in the same way, substituting this limit in the equality (19) of article 17°, we find the required work done :

$$T = \frac{2}{3} R^3 H.$$

24.° General Remarks. As a matter of fact all the above problems are solved, by one and the same method. This method is as follows : the quantity to be evaluated is divided into a large number of infinitesimal terms of the same type. These infinitesimal terms are calculated approximately. By increasing the number of terms, we get more exact expressions. Then the required quantity is evaluated by the summation of the calculated infinitesimal quantities. The result obtained for the required quantity is, however not exact, and in order to find the exact result, it is necessary to take the limit of the given sum by increasing the number of infinitesimal terms indefinitely.

In brief, the given method is to bring the quantity to be evaluated in the form of the limit of a sum of indefinitely increasing-number of (ultimately vanishing) terms, or, as it is more commonly stated, in the form of indefinitely large number of infinitesimal terms.

This method is one of the most important methods of higher mathematics ; it is studied in that branch of mathematics called "Integral Calculus." In this subject we deal with the limit of the sums of indefinitely increasing

number of ultimately vanishing terms. These limits are also called "Integrals". Thus, going through the procedures of the preceding articles, we can say that in each of them we had to calculate an integral.

The sums considered have a very simple form ; *viz*, they are the sums of the following types :

$$\sum_{k=1}^{n} k, \quad \sum_{k=1}^{n} k^2, \quad \sum_{k=1}^{n} k^3.$$

These were derived in chapter 1. When we came across a sum of a more complicated type like

$$\frac{1}{n^3} \sum_{k=1}^{n} k \sqrt{n^2-k^2} \,,$$

then we had to adopt an artificial method of finding its limit as was done in solving the example of article 22°. In integral calculus, general methods of finding the limits of sums of more complicated types have been expounded, and the solutions of similar problems are extremely facilitated.

Mathematicians did not find these general methods in a small span of time. On the contrary their establishment was the result of cumulative work of many generations. These methods were put forth in the modern form in the works of Leibnitz (1646—1716) and Newton (1642—1727). However the idea of expansion to indefinitely large number of infinitesimals was known long before them. Strictly speaking, the concept was already known to the mathematicians of ancient Greece (mainly to Archimedes, 287—212 B.C.).

Archimedes particularly, knew the volume of a sphere, a cone, their segments and also the volume of a cylindrical

segment.

In the medieval age the scientific thinking was relegated to the background and it is only from the beginning of the XVI century that the natural sciences, mathematics in particular, again began to develop. Scientists began to discover a fresh the results developed in ancient times and gradually extended them. This also holds true in the case of the method of summation of infinitesimals. This method received a significant advancement in the works of Kepler, "Solid geometry of wine barrels" (1615) and Cavalieri "Geometry of Indivisibles" (1635).

However in both these works the authors did not obtain general methods of determining the limits of sums or integrals. In this respect the methods adopted in this book very nearly correspond to the methods adopted by these two authors though ~~through~~ the exposition is essentially distinct.

Later, more general methods of solving the integrals were gradually found, and, as already stated, the problem was dealt by Leibnitz and Newton (the very term "integral" belongs to the school of Leibnitz and was introduced in 1690).

25.° **Principle of Cavalieri.** Not having been able to find the limits of the sums of complicated type, Cavalieri expounded a very useful principle, which in a number of cases, helped him to avoid the calculation of these limits. This principle is formulated as follows :

"If there are two bodies lying between parallel planes P and Q (Fig. 17) in such a way that they always intercept figures of equal areas when cut by any plane R parallel to P and Q, the volumes of these bodies are equal."

To prove this principle, let us draw $(n-1)$ planes parallel to P and Q. These planes cut both the bodies into n parts

and these parts are approximately cylindrical or prismatic, then we shall find that their volumes are equal having bases of equal areas and the same height. Hence their volumes are approximately equal. The greater the number of divisions, better the approximations. Hence, the result enunciated in the principle.

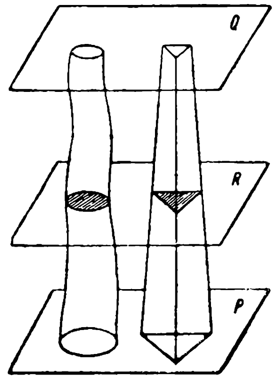

Fig. 17.

It is easy to generalise this principle showing that if the intersection of both the bodies are such that the areas of sections are in a certain ratio, the ratio of the volumes will be the same.

It is not difficult to establish a similar principle for the areas. The formulation of the principle in this case will be as follows : If two flat figures I and II lie between two parallel lines p and q (Fig. 18) in such a way that they intercept segments of equal length on any straight line r, parallel to p and q, then both the figures have the same areas.

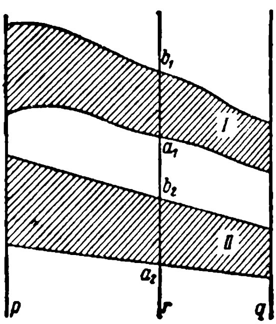

Fig. 18.

If the ratio of the segments a_1b_1 and a_2b_2 be equal to a number k, not depending on the position of the straight line r, then the ratio of the areas of figure I to the area of figure II is equal to k.

The proof of these statements can be arrived at by the reader in the same manner as was done in the case of the volumes.

CHAPTER V

§ 5. PARABOLA AND ELLIPSE

26. Area of the parabola. We shall consider the curve the equation of which in the rectangular system of co-ordinates is

$$y = ax^2 \tag{25}$$

This curve is called the parabola ; it is of the form as shown in figure 19 (we assume that $a > 0$). Let us take an arbitrary point M on the parabola and draw the perpendicular MP from it on the x-axis.

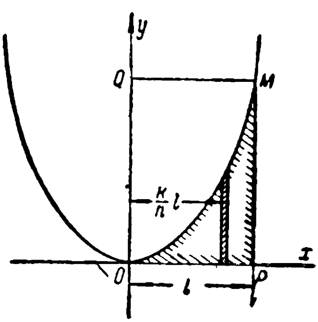

Fig. 19.

The problem is to find the area of the curvilinear triangle OMP.

In order to solve the problem, divide the segment OP into n equal parts and draw perpendiculars from the points of division to intersect the parabola. These perpendiculars cut the reguired area into n narrow vertical strips. All these strips can be supposed to be nearly rectangles. We shall calculate their areas.

Let us denote the whole length OP by l and consider the k-th strip. Its width is equal to $\frac{1}{n}l$. Its height is deter-

mined as follows: the distance of the strip from the y-axis is $\frac{k}{n}l$, and since its upper end lies on the parabola, the height of the strip is equal to the ordinate of the point on the parabola, represented by the equation (25) and is equal to

$$a\left(\frac{k}{n}l\right)^2.$$

Hence the area of the strip

$$al^3\frac{k^2}{n^3},$$

and the area of the whole triangle OMP is the sum

$$F \doteq \sum_{k=1}^{n} al^3\frac{k^2}{n^3},$$

or

$$F \doteq al^3 \cdot \frac{1}{n^3}\sum_{k=1}^{n} k^2,$$

or, finally,

$$F \doteq \frac{al^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right).$$

In order to get the exact area the number n must be increased indefinitely. In the limit, we find

$$F = \frac{al^3}{3}.$$

We can give this result a simple geometrical meaning. For instance, let us consider the rectangle $OQMP$. Its area is evidently equal to $OP.PM$. But $OP = l$; PM is the ordinate of the point M whose abscissa is l, so from the equation of the parabola, $PM = al^2$.

Therefore, the area $OQMP$ is al^3, and, consequently, the

area of the triangle OMP is equal to one third of the area of the rectangle $OQMP$. Hence the area of the triangle OQM is equal to two third of the area of the same rectangle.

These elegant results were first found by Archimedes.

The determination of an area is called 'Quadrature'. Thus, we have carried out the Quadrature of a parabola.

27°. The volume of paraboloid of revolution. Let us imagine that the parabola considered in the previous article revolves about the y-axis. (Fig. 20.) The surface which is described is called the "paraboloid of revolution." Let us consider a plane A, perpendicular to the axis of y. We shall determine the volume of the body bounded between the paraboloid and the plane.

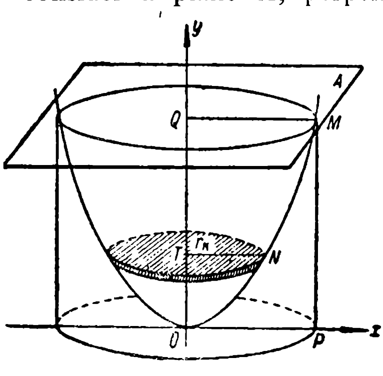

Fig. 20

In order to do this, we shall divide the segment OQ into n equal parts and pass through these points, planes parallel to the plane A. These planes cut the given body into n layers each of which, we suppose, is cylindrical. If the distance OP, be denoted by l, then, as proved earlier $OQ = al^2$. The height of each elementry cylinder is $\frac{1}{n} al^2$.

To determine the radius of the k-th—→ cylinder, we shall proceed as follows: if the radius be $r_k = NT$, it obviously

represents the abscissa of the point N on the parabola. Therefore, the ordinate of this point is

$$OT = \frac{k}{n} OQ = \frac{k}{n} al^2,$$

and from the equation of the parabola, we find

$$\frac{k}{n} al^2 = ar_k^2,$$

hence

$$r_k^2 = \frac{k}{n} l^2,$$

and

$$\pi r_k^2 = \pi l^2 \frac{k}{n}.$$

Hence the infinitesimal volume is

$$V_k \doteq a\pi l^4 \frac{k}{n^2}.$$

The required volume V is, then

$$V \doteq a\pi l^4 \cdot \frac{1}{n^2} \sum_{k=1}^{n} k,$$

from which, after simple calculation, we find

$$V \doteq a\pi l^4 \cdot \frac{1}{2}\left(1 + \frac{1}{n}\right).$$

Increasing n indefinitely, we find the exact value of the volume of paraboloid of revolution :

$$V = \frac{1}{2} \pi a l^4.$$

Let us compare this volume with the volume of the

cylinder with radius $R = OP$ and height $H = OQ$. Its volume is

$$\pi R^2 H = \pi (OP)^2 \cdot OQ$$
$$= \pi l^2 .\ a l^2 = \pi a l^4.$$

This leads us to the *theorem of Archimedes* :

The volume of the paraboloid of revolution is equal to half the volume of the cylinder with the same base and the same height.

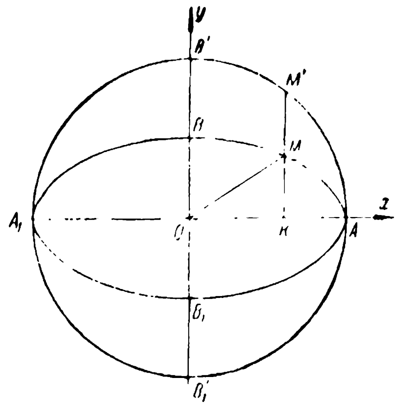

Fig. 21.

28°. Ellipse and its area. We shall consider a very important curve called the 'ellipse'. It may be defined as a compressed circle. Now we shall explain this definition.

Let us consider a circle with some radius 'a'. We shall assume that it lies in a plane of a system of rectangular co-ordinates and its centre coincides with the origin (Fig. 21). Further let the ordinates KM' of all the points M' on the circumference be compressed to the extent that the coefficient of compression is $q < 1$, *i.e.*

$$KM : KM' = q.$$

This operation of compression transforms the circle $AB'A_1B'_1$, into an ellipse. We shall find the equation of the ellipse. If the coordinates of the point M on the ellipse be (x, y), we find, by the definition of the ellipse :

$$y = q \cdot KM'.$$

But by the theorem of Pythagoras

$$KM' = \sqrt{(OM')^2 - (OK)^2} = \sqrt{a^2 - x^2},$$

so that

$$y = q\sqrt{a^2 - x^2}.$$

If we denote OB by b, then by the definition of the ellipse we have

$$b : a = OB : OB' = q,$$

so that

$$q = \frac{b}{a},$$

and the equation of the ellipse takes the form :

$$y = \frac{b}{a}\sqrt{a^2 - x^2},$$

hence

$$\frac{y^2}{b^2} = \frac{1}{a^2}(a^2 - x^2)$$

and finally

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

It is called the "Canonical" or the "Simplest" equation of the ellipse.

Next we shall determine the area of the ellipse. Making use of the principle of Cavalieri given at the end of the article 25°, we can at once say, that the ratio of the area of the ellipse to the area of the circle is equal to the coefficient of compression 'q', so that, denoting the area of the ellipse by F, we have

$$F : \pi a^2 = q,$$

or

$$F = q\pi a^2.$$

Substituting the value $q = \frac{b}{a}$, we finally get

$$F = \pi ab.$$

Making use of the principle of Cavalieri, we can also easily find the volume of revolution of the ellipse about the x-axis. (Fig. 22), as the ratio of the radii of circles formed by the intersection of the ellipsoid by the planes, perpendicular to the x-axis, to the radii of intersection of the sphere by the same planes, is equal to q (Fig. 22). Therefore, the ratio of their areas is equal to q^2.

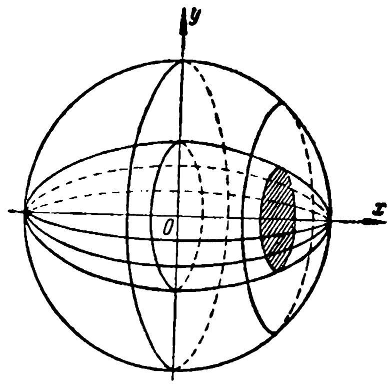

Fig. 22.

By the principle of Cavelieri, the same is the ratio of the two volumes, which gives,

$$V : \frac{4}{3}\pi a^3 = q^2 = \frac{b^2}{a^2},$$

so that

$$V = \frac{4}{3}\pi ab^2.$$

CHAPTER VI

§6. SINUSOID

29. About a trigonometrical sum. In this chapter, we shall need the expressions for the following sum :

$$S = \sum_{k=1}^{n} \sin k\alpha = \sin \alpha + \sin 2\alpha + \ldots + \sin n\alpha, \qquad (26)$$

where α is some definite angle.

In order to find this sum we multiply both sides of the equality (26) by $2\sin\frac{\alpha}{2}$:

$$2S \sin \frac{\alpha}{2} = 2 \sin \alpha \sin \frac{\alpha}{2} + 2 \sin 2\alpha \sin \frac{\alpha}{2} + \\ + \ldots + 2 \sin n\alpha \sin \frac{\alpha}{2}$$

and apply to each term on the right-hand-side the important formula $2 \sin A \sin B = \cos (A-B) - \cos (A+B)$.

This gives

$$2S \sin \frac{\alpha}{2} = \left[\cos \frac{\alpha}{2} - \cos \frac{3}{2}\alpha\right] + \left[\cos \frac{3}{2}\alpha - \cos \frac{5}{2}\alpha\right] + \\ + \ldots + \left[\cos \frac{2n-1}{2}\alpha - \cos \frac{2n+1}{2}\alpha\right].$$

It is easy to see that the first term of each bracket (except the first term itself) cancels with the second term of the preceding bracket. Hence

$$2S \sin \frac{\alpha}{2} = \cos \frac{\alpha}{2} - \cos \frac{2n+1}{2} \alpha. \tag{27}$$

Applying the important formula

$$\cos A - \cos B = 2 \sin \frac{B-A}{2} \sin \frac{A+B}{2},$$

we write (27) in the form

$$2S \sin \frac{\alpha}{2} = 2 \sin \frac{n\alpha}{2} \sin \frac{(n+1)\alpha}{2},$$

hence

$$S = \frac{\sin \frac{n\alpha}{2} \sin \frac{(n+1)\alpha}{2}}{\sin \frac{\alpha}{2}}$$

So

$$\sum_{k=1}^{n} \sin k\alpha = \frac{\sin \frac{n\alpha}{2} \sin \frac{(n+1)\alpha}{2}}{\sin \frac{\alpha}{2}}. \tag{28}$$

This is the required formula.

30°. Auxiliary inequality. Let α be an arbitrary angle *), satisfying the condition $0 < \alpha < \frac{\pi}{2}$. Then

$$\tan \alpha > \alpha > \sin \alpha. \tag{29}$$

In order to prove this statement, we shall refer to the figure 23. From it, we can intuitively find that the triangle

*) to be more specific the value of angle α is given in radiaus.

OCA is contained wholly in the sector OCA, which, in its turn, is fully enclosed in the triangle OAB. It can be seen, from the areas of the figures, the following inequality holds :

area of $\triangle\, OAB >$

$>$ area of the sector $OCA >$

$>$ area of $\triangle OCA$.

In other words,

$$\frac{1}{2} OA \cdot AB > \frac{1}{2} R \cdot \overset{\frown}{CA} > \frac{1}{2}\, OA \cdot CD.$$

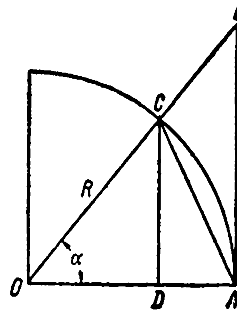

Fig. 23.

But

$$OA = R,\ AB = R \tan \alpha,$$
$$\overset{\frown}{CA} = R\alpha,\ CD = R \sin \alpha,$$

so that

$$\frac{1}{2} R^2 \tan \alpha > \frac{1}{2} R^2\alpha > \frac{1}{2} R^2 \sin \alpha.$$

Cancelling the positive factor $\frac{1}{2} R^2$ in this double inequality, we get the inequality (29).

31°. Sine of infinitesimal angle. Let us assume that angle α approaches zero, successively through the values $\alpha_1, \alpha_2, \alpha_3, \ldots$

In this case the formula

$$\boxed{\lim \frac{\sin \alpha_n}{\alpha_n} = 1} \tag{30}$$

holds good, which is one of the most important formulae in mathematics.

In order to prove this formula, we may suppose that all the values α_n are positive, since, the value of the expression $\frac{\sin \alpha_n}{\alpha_n}$ does not change when we replace α_n by $-\alpha_n$. Besides we may consider that $\alpha_n < \frac{\pi}{2}$, for, in all cases, this is so

for sufficiently large values of n.

Hence,

$$0 < \alpha_n < \frac{\pi}{2},$$

and then, by (29),

$$\tan \alpha_n > \alpha_n > \sin \alpha_n,$$

and dividing all the parts of the inequality by the positive quantity $\sin \alpha_n$, we get,

$$\frac{1}{\cos \alpha_n} > \frac{\alpha_n}{\sin \alpha_n} > 1.$$

Taking the reciprocals of these quantities and reversing the inequalities, we get

$$\cos \alpha_n < \frac{\sin \alpha_n}{\alpha_n} < 1. \tag{31}$$

As α_n approaches zero (as it is easily seen from the figure) the cosine of this angle approaches unity : *i. e.*

$$\lim [\cos \alpha_n] = 1,$$

and since [by (31)], the fraction $\frac{\sin \alpha_n}{\alpha_n}$ lies between 1 and $\cos \alpha_n$, it must also approach unity, which proves the formula (30).

32°. **Quadrature of sinusoid.** We shall consider the curve having the equation

$$y = \sin x. \tag{32}$$

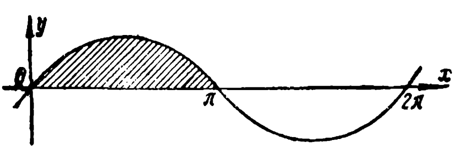

Fig. 24.

Its graph is given in figure 24 and is called the sinusoid.

We shall find the area of the figure enclosed by the part

of the sinusoid from $x = 0$ to $x = \pi$ and the x-axis (this being the shaded area in figure 24).

To obtain this, as usual, divide the segment on the x-axis from $x = 0$ to $x = \pi$ into n parts by the points

$$x_1 = \frac{\pi}{n}, \; x_2 = \frac{2\pi}{n}, \; \ldots, \quad x_n = \frac{n\pi}{n}$$

and draw perpendiculars from these points to cut the sinusoid. The length of the perpendiculars is found from the equation (32) and are equal to

$$\sin\frac{\pi}{n}, \quad \sin\frac{2\pi}{n}, \quad \sin\frac{3\pi}{n}, \; \ldots, \; \sin\frac{n\pi}{n}$$

(the last of them is equal to zero). These perpendiculars divide the whole area into n strips of width $\frac{\pi}{n}$. Assuming each of these strips as rectangles with base $\frac{1}{n}\pi$ and height (for the k-th strip), equal to $\sin\frac{k\pi}{n}$, we shall have an approximate expression for the area of the k-th strip

$$F_k \doteq \frac{\pi}{n}\sin\frac{k\pi}{n}.$$

Hence the required area is approximately equal to

$$F \doteq \frac{\pi}{n}\sum_{k=1}^{n}\sin\frac{k\pi}{n}.$$

This expression, on the basis of the formula (28) of article 29°, putting $\alpha = \frac{\pi}{n}$, can be written as:

$$F \doteq \frac{\pi}{n} \cdot \frac{\sin \frac{\pi}{2} \cdot \sin \frac{(n+1)\pi}{2n}}{\sin \frac{\pi}{2n}},$$

or $\left(\text{because } \sin \frac{\pi}{2} = 1\right)$

$$F \doteq \frac{\pi}{n} \cdot \frac{\sin \frac{(n+1)\pi}{2n}}{\sin \frac{\pi}{2n}}. \tag{33}$$

The exact expression for the given area is the limit of the right-hand-side of the equality (33) when n increases indefinitely. This limit is found by the following consideration.

Evidently

$$\frac{(n+1)\pi}{2n} = \frac{\pi}{2} + \frac{\pi}{2n},$$

so that this angle approaches $\frac{\pi}{2}$, and, therefore, as it is easily seen from the figure, the sine of this angle must approach unity :

$$\lim \sin \frac{(n+1)\pi}{2n} = 1. \tag{34}$$

In the denominator angle $\alpha_n = \frac{\pi}{2n}$ approaches zero and therefore, by the formula 30 of article 31°,

$$\lim \left(\frac{\pi}{n} \quad \frac{1}{\sin \frac{\pi}{2n}} \right) = \lim \left[2 \quad \frac{\alpha_n}{\sin \alpha_n} \right] = 2. \tag{35}$$

From (34) and (35) (by the theorem on the limit of a product), we finally find,

$$F = 2.$$

So, the area, bounded by half the arc of the sinusoid and by half the chord is equal to 2.

33°. Volume of the solid of revolution of the sinusoid. We shall assume that the sinusoid, as drawn in figure 24, revolves about the x-axis. We shall now find the volume V of the body, bounded by the surface formed by the revolution of half the arc of the sinusoid.

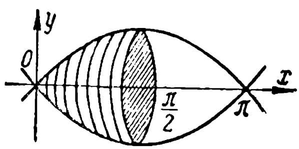

Fig. 25.

In order to do this we shall draw a plane through $x = \frac{\pi}{2}$, perpendicular to the x-axis. It is evident, this plane (Fig. 25) cuts the body into two equal parts; we shall find the volume V^* of the left half of the given body. Divide the segment, on the axis of x, between $x = 0$ and $x = \frac{\pi}{2}$ into n equal parts, by the points,

$$x_k = k\frac{\pi}{2n} \qquad (k=1, 2, \ldots, n).$$

Draw planes through these points perpendicular to the x-axis. Considering the elementary layer lying between $(k-1)$-th and k-th planes as a cylinder of radius r_k and height $h = \frac{\pi}{2n}$, we find the elementary volume

$$V_k = \pi r_k^2 h = \frac{\pi^2}{2n}\sin^2\frac{k\pi}{2n},$$

hence the toral volume of the left half of the body is approximately equal to

$$V^* \approx \frac{\pi^2}{2n}\sum_{k=1}^{n}\sin^2\frac{k\pi}{2n}.$$

The exact value for the volume is the limit of this expression when n increases indefinitely :

$$V^* = \lim\left[\frac{\pi^2}{2n}\sum_{k=1}^{n}\sin^2\frac{k\pi}{2n}\right]. \tag{36}$$

To determine this limit we shall apply a method, which considerably shortens the calculation.

The method is as follows : we shall consider side by side with the sinusoid (32), the curve, which is a graph of the function

$$y = \cos x. \tag{37}$$

Since

$$\cos x = \sin\left(x + \frac{\pi}{2}\right),$$

we easily find that the curve (37) is the same as the sinusoid (32), but shifted along the x-axis to $\frac{\pi}{2}$ on the left (Fig. 26).

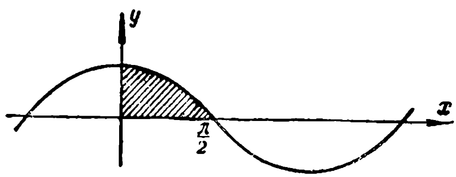

Fig. 26

We shall now suppose that we revolve this sinusoid about the x-axis. It is clear, that the volume of the body, formed by the revolution of the portion shaded in the figure 26, is equal to the volume of the left half of the original body (it exactly coincides with the volume of the right half of the given body).

On the other hand, had we calculated this volume by the method of summation, then, evidently, we would have found

the limit, as in (36), by substituting all the sines by cosines *i. e.* we would have got

$$V^* = \lim\left[\frac{\pi^2}{2n}\sum_{k=1}^{n}\cos^2\frac{k\pi}{2n}\right]. \tag{38}$$

Thus the same quantity V^* has been written in two ways :

$$V^* = \lim\left[\frac{\pi^2}{2n}\sum_{k=1}^{n}\sin^2\frac{k\pi}{2k}\right] = \lim\left[\frac{\pi^2}{2n}\sum_{k=1}^{n}\cos^2\frac{k\pi}{2n}\right].$$

Adding both the expressions (which is, evidently, permissible under the sign of limit), we get :

$$2V^* = \lim\left[\frac{\pi^2}{2n}\sum_{k=1}^{n}\left(\sin^2\frac{k\pi}{2n} + \cos^2\frac{k\pi}{2n}\right)\right]. \tag{39}$$

But

$$\sin^2\alpha + \cos^2\alpha = 1,$$

so that each term in the sum (39) is equal to unity, and since the number of terms is n,

$$2V^* = \lim\left[\frac{\pi^2}{2n}\cdot n\right] = \lim\left(\frac{\pi^2}{2}\right) = \frac{\pi^2}{2}$$

for the constant quantity is itself equal to its limit.

The volume V^* of the left half of the body can be found by dividing by 2, but since, we are, from the very beginning interested in the volume of the whole body, the result is

$$V = 2V^* = \frac{\pi^2}{2}. \tag{40}$$

So the volume of the body bounded by the surface,

formed by the revolution of half the arc about its chord is equal to $\frac{\pi^2}{2}$.

°**34. Mean values.** Let some quantity y take a finite number of values :

$$y_1, y_2, y_3, \ldots, y_n.$$

In this case, the Arithmetic Mean

$$y_* = \frac{y_1+y_2+\ldots+y_n}{n}$$

of the number y_k, is called the *Mean Value* of the quantity y. The usefulness of considering this value is in two of its properties.

A. If all the values of the quantity y lie between the numbers m and M, then the mean value also lies between these numbers, *i. e.* if

$$m \leqslant y_k \leqslant M \quad (k = 1, 2, \ldots, n), \tag{41}$$

then

$$m \leqslant y_* \leqslant M.$$

B. If all the values of the quantity y be equal to one and the same number h, then the mean value is also equal to that number.

The property **B** is evident, and for the proof of the property **A**, we should add all the inequalities of (41), which gives,

$$nm \leqslant \sum_{k=1}^{n} y_k \leqslant nM,$$

and then divide this inequality by n.

Along with the mean value y_* of the quantity y, we often need the root mean square y^* of this quantity. This mean is defined by the equality

$$y^* = \sqrt{\frac{y_1^2 + y_2^2 + \ldots + y_n^2}{n}}. \tag{42}$$

In other words, the root mean square of the quantity y is the square root of the mean value of the quantity y^2.

It is easy to show that if each of the values of y is not negative, then their mean square possesses the properties **A** and **B** of the Arithmetic Mean.

In fact, if

$$0 \leqslant m \leqslant y_k \leqslant M \quad (k = 1, 2, \ldots, n),$$

then

$$m^2 \leqslant y_k^2 \leqslant M^2 \quad (k = 1, 2, \ldots, n).$$

Adding all these inequalities and dividing the result by n and taking the square root, we get,

$$m \leqslant y^* \leqslant M.$$

which proves that y^* possesses the property **A**. Property **B** is evident.

In the case considered above the quantity y was given finite number of values. In many practical problems we have to consider quantities which change continuously. For the calculation of mean values of such quantities it is necessary to apply the method of summation of infinitesimals. We shall illustrate this by an example in physics.

35°. The effective strength of a current. We shall consider a variable sinusoidal current

$$I = A \sin t, \tag{43}$$

where t is the time, I is the strength of the current and A a constant. At different times the quantity I has different values, its maximum value being A;

$$I_{\max} = A. \tag{44}$$

In electrical engineering, the root mean square I_e of the strength of the current for the time equal to the period of oscillation *i. e.* for the interval $t = 0$ to $t = 2\pi$, plays an important role.

It is observed, that in measuring the strength of the current by an ammeter, it indicates the quantity I_e. This quantity is called the effective strength.

We shall now calculate I_e for the current (43).

For this we divide the interval of time from $t = 0$ to $t = 2\pi$ into n small intervals by the points

$$t_k = \frac{2\pi}{n} k \quad (k = 1, 2, \ldots, n).$$

If the number n is very great, then it is possible to consider that for the time interval t_{k-1} to t_k, the strength of the current does not change and is approximately equal to its value at the moment t_k, *i.e.*

$$I_k \doteq A \sin \frac{2\pi}{n} k.$$

In other words, we assume that the current during the infinitesimal interval of time is constant. By this simplified assumption, the effective current strength will be equal to

$$I_e \doteq \sqrt{\frac{\sum_{k=1}^{n} I_k^2}{n}} = \sqrt{\frac{A^2 \sum_{k=1}^{n} \sin^2 \frac{2\pi}{n} k}{n}} \tag{45}$$

The real value of I_e is the limit of the right-hand-side of the equality (45) when n increases indefinitely :

$$I_e = \lim \left[A \cdot \sqrt{\frac{1}{n} \sum_{k=1}^{n} \sin^2 \frac{2\pi}{n} k} \right].$$

We shall find the limit of the expression under the square-root;

$$\lim \left[\frac{1}{n} \sum_{k=1}^{n} \sin^2 \frac{2\pi}{n} k \right]. \tag{46}$$

This can be done easily as follows :

Let us suppose, we wish to find the volume of the body formed by the revolution of one wave of the sinusoid (32) about the x-axis. If we apply the method of summation of the infinitesimals, then repeating the process of article 33°, we find this volume in the form :

$$\lim \left[\frac{2\pi^2}{n} \sum_{k=1}^{n} \sin^2 \frac{2k\pi}{n} \right].$$

On the other hand, this volume is, evidently, twice as much as the volume of the body described by the revolution of the semi-arc of the sinusoid *i. e.* the required volume is equal to π^2. Hence

$$\lim \left[\frac{2\pi^2}{n} \sum_{k=1}^{n} \sin^2 \frac{2k\pi}{n} \right] = \pi^2. \tag{47}$$

It is easily seen that the limit (46) is derived from the limit (47) by dividing by $2\pi^2$; *i. e.*

$$\lim \left[\frac{1}{n} \sum_{k=1}^{n} \sin^2 \frac{2k\pi}{n} \right] = \frac{1}{2}.$$

In this case *)

$$I_e = A \sqrt{\frac{1}{2}}. \tag{48}$$

*) Here we make use of the theorem : if the variable quantity $x_n \geqslant 0$ approaches the limit a, then $\sqrt{x_n}$ approaches $\sqrt{a}$.

If we compare the formulae (48) and (44), we see that

$$I_{\max} = I_e \sqrt{2},$$

i. e. the maximum strength of the current is approximately one and a half times greater than that indicated by the ammeter.

EXERCISES

We give here a few problems in the methods discussed. We advise the readers to work out these exercises. Newton has rightly said "In Mathematics, examples are more useful than the rules".

1) Find the sum

$$S_4 = \sum_{k=1}^{n} k^4.$$

Answer $S_4 = \dfrac{n\,(n+1)\,(2n+1)\,(3n^2+3n-1)}{30}$.

2) Find the area of the right-angled triangle by the method of summation.

3) Find the area bounded by the x-axis, the curve $y = x^3$ and the straight line $x = 1$.

Answer $\dfrac{1}{4}$.

4) Find $\lim \left[\dfrac{1}{n^2} \sum_{k=1}^{n} \sqrt{n^2-k^2} \right]$ when n increases indefinitely

Hint : determine the area of one quarter of a circle by the summation of rectangular strips.

Answer $\frac{\pi}{4}$.

5) Proceeding from the result of the previous question, find the volume of a cylinder, dividing it into rectangular strips as shown in figure 9.

6) Find the pressure of water on the walls of a cylindrical glass vessel.

Answer $P = \pi RH^2$.

7) Find the work done in pumping out water from a conical vessel whose base is horizontal and lies below the vertex.

Answer $T = \frac{1}{2} \pi R^2 H^2$.

8) Find the volume of an ellipsoid of revolution by direct calculation without any reference to the principle of Cavalieri.

9) Find the volume of ellipsoid formed by the revolution of the ellipse about the y-axis.

Answer $V = \frac{4}{3} \pi a^2 b$.

10) How much work is done in pumping out water from a hemisphere placed on its diametrical plane.

Answer $T = \frac{5}{12} \pi R^4$.

11) Find the pressure of water on the walls of a prismatic vessel of height H with the perimeter of the base equal to p.

Answer $P = \frac{1}{2} pH^2$.

12) On the basis of the result of exercise 1, find the volume of a body bounded by the surface formed by the revolution of the parabola $y = ax^2$ about x-axis and the plane perpendicular to the x-axis at a distance h from the origin

Answer. $V = \frac{1}{5} \pi a^2 h^5$.

13) Find the limit

$$\lim \left[\frac{1}{n^{p+1}} \sum_{k-1}^{n} k^p \right]$$

when n increases indefinitely (p is a natural number).

Answer. $\frac{1}{p+1}$.

14) Find the limit

$$\lim \left[\frac{1}{n} \sum_{k=1}^{n} \sqrt{\frac{k}{n}} \right]$$

when n increases indefinitely.

Hint: find the area of the curvilinear triangle OQM (Fig. 19), dividing it into strips parallel to x-axis.

Answer. $\frac{2}{3}$.

15) Find the sum

$$S = \sum_{k=1}^{n} \cos ka.$$

$$\text{Answer. } S = \frac{\sin \frac{2n+1}{2} a}{2 \sin \frac{\alpha}{2}} - \frac{1}{2}.$$

16) Making use of the preceding result, find the area F of the figure bounded by the curve $y = \cos x$ and the axes of co-ordinates.

Answer. $F = 1$.